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From noncommutative bosonization to S-duality

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Abstract: We extend standard path-integral techniques of bosonization and duality to the setting of noncommutative geometry. We start by constructing the bosonization prescription for a free Dirac fermion living in the noncommutative plane $\mathbb{R}^2_\theta$. We show that in this abelian situation the fermion theory is dual to a noncommutative Wess-Zumino-Witten model. The non-abelian situation is also constructed along very similar lines. We apply the techniques derived to the massive Thirring model on noncommutative $\mathbb{R}^2_\theta$ and show that it is dualized to a noncommutative WZW model plus a noncommutative cosine potential (like in the noncommutative Sine-Gordon model). The coupling constants in the fermionic and bosonic models are related via strong-weak coupling duality. This is thus an explicit construction of S-duality in a noncommutative field theory.

Keywords: Field Theories in Lower Dimensions, Non-Commutative Geometry.

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Contents

1. Introduction and discussion

Quantum field theories on noncommutative spaces has been a subject of renewed interest since the recent discovery of its connections to string and M theories, see, e.g., [1]–[5] and references therein. From a string theory point of view, it was realized in these works that one can translate the effects of a large background magnetic field into a deformation of the D-brane world-volume. Still, one can envisage studying such theories from a purely quantum field theoretic point of view. For example, perturbative aspects of such noncommutative field theories have been studied and have revealed a surprising mixing of the IR and the UV [6, 7, 8]. These phenomena are directly related to the string theoretic origins of these theories, but one would also like to know to which extent properties of quantum field theories on commutative spaces also arise in quantum field theories on noncommutative spaces. This may be of some interest given that it is not always simple to extract quantum results from string theory, while we are used to do so in field theory.

One important feature of many conventional quantum field theories is that of duality. As an example, it follows from bosonization in $1+1$ dimensions [9, 10] that the Sine-Gordon model of a single scalar field,

$$\int d^2x \left\{ \frac{1}{2}\partial_\mu \phi \partial^\mu \phi + \frac{\alpha_0}{\beta^2}(\cos \beta\phi - 1) \right\}, \tag{1.1}$$

is dual to the massive Thirring model of a fermion field,

$$\int d^2x \left\{ \bar{\psi}(i\gamma^\mu \partial_\mu + m)\psi - \frac{\lambda}{2} j_\mu j^\mu \right\}. \tag{1.2}$$

JHEP07(2000)030

One can relate bosonic composite operators to fermionic ones and vice-versa using the standard bosonization machinery. Of particular interest to us in here is that the Sine-Gordon/Thirring model duality is a strong/weak coupling duality since the coupling constants of the two theories are related according to:

$$\frac{4\pi}{\beta^2} = 1 + \frac{\lambda}{\pi}\,. \tag{1.3}$$

The purpose of this paper is to study the analog of this duality on a noncommutative spacetime. In order to do this, we begin by considering bosonization on the noncommutative plane and will see how the bosonization rules get generalized to this situation. This is done in section 2, where we study the abelian bosonization of a free fermion field in two noncommuting dimensions, employing path-integral techniques [11]–[14]. We shall learn that the free fermion action is bosonized to a noncommutative U(1) WZW-action. That the WZW term in the action is nonvanishing for a U(1) valued field is simply due to the noncommutativity of spacetime. In fact, the procedure follows much as for the conventional non-abelian bosonization [15], and the rules are very similar both in the non-abelian and noncommutative cases. Because of this, the non-abelian noncommutative bosonization will be a simple standard extension of the abelian noncommutative bosonization. In particular, the non-abelian free fermion action bosonizes to a noncommutative U(N) WZW model.

A question that immediately arises is the following. In the abelian case, the free fermion has a quadratic kinetic action and therefore noncommutative and commutative descriptions should match. On the other hand the commutative abelian fermion is dualized to a free scalar field theory, apparently very different from a noncommutative U(1) WZW model. The same phenomena happens in the non-abelian situation, where the commutative abelian fermion is dualized to a commutative U(N) WZW model, again not the same as a noncommutative non-abelian WZW model. As we shall see, the noncommutativity in the free fermion action makes its appearance when we gauge the global symmetries in order to implement the path-integral duality techniques of [12, 13].

The fact that this happens raises some interesting possibilities for future research. Indeed this is apparently establishing some sort of relation between commutative and noncommutative WZW models, and one could interpret this as a different version of the Seiberg-Witten map between commutative and noncommutative descriptions of the Born-Infeld action [4]. It would be very interesting to find a string theory realization of this field theoretic scenario, and try to understand this relation between WZW models from a kind of B-field point of view.

After understanding the free fermion we proceed in section 3 to interacting theories, with the goal of realizing S-duality for noncommutative field theories. We shall see that the massive Thirring model on noncommutative space is dual to a WZW model plus a noncommutative cosine potential. The usual relation (1.3) between the coupling constants of the dual theories continues to hold in the noncommutative case, thus realizing an example of S-duality. Observe that in here the knowledge of the bosonization rules for the noncommutative free fermion plays a central role, as they allow us to derive the noncommutative duality in very simple steps. Indeed they allow for a full and explicit quantum construction of S-duality in this noncommutative setting.

Understanding duality in noncommutative field theory, one could hope to gain some starting grounds in order to try to match these results to a string theory description. This is not a clear task, however. On one hand the results in [4] are derived at the CFT disk level, so that one can not assume that an S-duality in noncommutative field theory will translate to a string theory S-duality. On the other hand, we are dealing with simple bosonic theories which do not have immediate brane realizations. After this paper was concluded, two pre-prints appeared that describe S-duality in noncommutative gauge theories [16, 17], and therefore have a closer connection to the string theory description [4]. What we would like to stress from our work is that not only it provides a construction which does not rely on any string theory connection, but it also allows for an explicit and exact treatment.

Some words on notation. To study noncommutative bosonization in $\mathbb{R}^2_\theta$, the underlying $\mathbb{R}^2_\theta$ will be labeled by noncommuting coordinates satisfying $[x^\mu, x^\nu] = i\theta^{\mu\nu}$. Here $\theta^{\mu\nu}$ is real and antisymmetric and so in two dimensions one has $\theta^{\mu\nu} = \theta\epsilon^{\mu\nu}$. The algebra of functions on noncommutative $\mathbb{R}^d_\theta$ can be viewed as an algebra of ordinary functions on the usual $\mathbb{R}^d$ with the product deformed to the noncommutative, associative star product,

$$(\phi_1 \star \phi_2)(x) = e^{\frac{i}{2}\theta^{\mu\nu}\partial^y_\mu\partial^z_\nu}\phi_1(y)\phi_2(z)\Big|_{y=z=x} . \tag{1.4}$$

Thus, we shall study theories whose fields are functions on ordinary $\mathbb{R}^2$, with actions of the usual form $S = \int d^2x\mathcal{L}[\phi]$, except that the fields in $\mathcal{L}$ are multiplied using the star product. Moreover, for any noncommutative theory the quadratic part of the action is the same as in the commutative theory, since if f and g are functions that vanish rapidly enough at infinity,

$$\int d^dx f \star g = \int d^dx f g . \tag{1.5}$$

2. Bosonization on noncommutative space

In this section we derive the bosonization rules of the free fermion action on a two-dimensional noncommutative space [18], using the path-integral approach described in [13, 14]. Our derivation is carried out for the abelian case (and will largely follow [14]) but as we shall see it generalizes immediately to the non-abelian case [15]. In [15] it was shown that a conventional free fermionic theory in $1 + 1$ dimensions is equivalent to a bosonic theory which is the WZW model. In our case the free fermionic theory is equivalent to a noncommutative version of the WZW model, even in the abelian case.

2.1 The noncommutative WZW model

Before looking at our specific problem, we start by taking the usual WZW model and define the noncommutative extension in the obvious way:

$$S[g] = \frac{1}{8\pi}\int_\Sigma d^2x\ (\partial_\mu g \star \partial_\mu g^{-1}) - \frac{i}{12\pi}\int_B d^3x \epsilon^{ijk}\ \left(g^{-1}\star\partial_i g\star g^{-1}\star\partial_j g\star g^{-1}\star\partial_k g\right). \tag{2.1}$$

If one would like to extend this action to the non-abelian situation, one simply needs to include a trace over the algebra. This theory has been discussed in a number of recent papers [19, 20, 21]. The manifold Σ is parametrized by (x^0, x^1) and is the boundary of the three-dimensional manifold B: $\partial B = \Sigma$. The $\star$-product on B is the trivial extension of the product on Σ, i.e. the extra dimension x^2 is taken to be commutative. The commutative non-abelian WZW model obeys the important Polyakov-Wiegmann identity [22, 23]:

$$S[gh^{-1}] = S[g] + S[h^{-1}] - \frac{1}{4\pi}\int_\Sigma d^2x\ \mathrm{Tr}(g^{-1}\partial_+ g h^{-1}\partial_- h)\,. \tag{2.2}$$

The same identity holds with a $\star$-product in the noncommutative case since the identity follows from using the cyclic property of the integral $\int A\star B = \int B\star A$ and from $g\star g^{-1} = 1$. Here the group element g of "noncommutative" U(1) is

$$g = e_\star^{i\alpha} = 1 + i\alpha - \frac{1}{2}\alpha\star\alpha - \frac{i}{6}\alpha\star\alpha\star\alpha + \cdots\,. \tag{2.3}$$

In the abelian commutative case, where $g = e^{i\alpha}$ without any $\star$-product, the action (2.1) of course reduces trivially to a free boson action $\int d^2x\ (\partial\alpha)^2$. However, the WZW action is nontrivial even for the abelian noncommutative case, where $g = e_\star^{i\alpha}$.

2.2 Definition of the current

Using the path-integral approach to bosonization [14, 13] one starts with the partition function of the free (abelian) fermion theory:

$$Z = \int \mathcal{D}\bar\psi\mathcal{D}\psi e^{-\int \bar\psi\star i\partial\!\!\!/\psi}\,. \tag{2.4}$$

We want to show that this theory is equivalent to a given bosonic theory. To prove that, one needs to show that the correlation functions obtained from the two theories are equal. We therefore consider the generating functional for these correlators as,

$$Z[s] = \int \mathcal{D}\bar\psi\mathcal{D}\psi e^{-\int \bar\psi\star(i\partial\!\!\!/+s\!\!\!/)\star\psi}\,, \tag{2.5}$$

where s_μ is an external source. Due to the noncommutativity of our problem, one might argue on where should one insert the source term, as for instance $\bar\psi \star \not{s} \star \psi \neq \not{s} \star \bar\psi \star \psi$. As we shall see later, (2.5) is the definition we need in order to be able to carry out the dualization procedure using the results for the fermionic determinant in [19]. So, we need to address the question of whether (2.5) is generating the correct conserved noncommutative current.

In the commutative fermi theory, the corresponding current is simply $j^\mu = \bar\psi\gamma^\mu\psi$ and this is conserved because of the equations of motion, which are $\not{\partial}\psi = 0$ and $\partial_\mu\bar\psi\gamma^\mu = 0$. In the noncommutative case one finds similarly that the conserved current is:

$$j^\mu = \bar\psi \star \gamma^\mu\psi = \bar\psi\gamma^\mu\psi + \frac{i}{2}\theta^{\alpha\beta}\partial_\alpha\bar\psi\gamma^\mu\partial_\beta\psi + \cdots . \tag{2.6}$$

Now consider the following source-term in the partition function:

$$\int d^2x \ \bar\psi \star \not{s} \star \psi, \tag{2.7}$$

where s_μ is a source. It is not immediately obvious that this will generate the correct correlation function, i.e. that

$$\frac{\delta}{\delta s_\mu(x)} Z[s] = \langle j^\mu(x)\rangle , \tag{2.8}$$

because of the infinite number of derivatives in the Moyal product. Let us see that this is actually true. Because of (1.5) one can remove for example the last $\star$-product in (2.7) and then this integral equals,

$$\int d^2x \left(e^{\frac{i}{2}\theta^{\mu\nu}\partial^y_\mu\partial^z_\nu}\,\bar\psi(y)\,\not{s}(z)\Big|_{y=z=x}\right)\psi(x) . \tag{2.9}$$

The first order term in θ can be written as,

$$\frac{i}{2}\theta^{\mu\nu}\partial_\mu\bar\psi\partial_\nu\not{s}\psi = \frac{i}{2}\theta^{\mu\nu}\partial_\nu[\partial_\mu\bar\psi\not{s}\psi] - \frac{i}{2}\theta^{\mu\nu}\partial_\mu\bar\psi\not{s}\partial_\nu\psi . \tag{2.10}$$

The first term on the r.h.s. is a total derivative and so vanishes under the integral sign. The second term on the r.h.s. seems to have the wrong sign (and the same appears to be the case with all higher-order odd terms in θ), since taking the functional derivative of the partition function with respect to s_μ will not lead to the current in (2.6). However, the second term in (2.10) also vanishes identically because of antisymmetry of $\theta^{\mu\nu}$ and because of the following identity for Dirac fermions:

$$\bar\chi\gamma^\mu\psi = (\bar\psi\gamma^\mu\chi)^\dagger , \tag{2.11}$$

which ensures that the commutative current is real. It will also ensure that the noncommutative current (2.6) is real, namely,

$$-\frac{i}{2}\theta^{\mu\nu}\partial_\mu\bar\psi\not{s}\partial_\nu\psi = -\frac{i}{2}\theta^{\mu\nu}(\partial_\nu\bar\psi\not{s}\partial_\mu\psi)^\dagger = \frac{i}{2}\theta^{\mu\nu}(\partial_\mu\bar\psi\not{s}\partial_\nu\psi)^\dagger = \frac{i}{2}\theta^{\mu\nu}\partial_\mu\bar\psi\not{s}\partial_\nu\psi , \tag{2.12}$$

JHEP07(2000)030

with the source s_μ being real, and where in the last equality we used the fact that the commutative current is real and one can therefore remove the dagger sign. All higher-order terms with odd number of θ's vanish for the same reason, e.g. the third order terms is — up to total derivatives — of the form,

$$\theta^{\mu\nu}\theta^{\alpha\beta}\theta^{\gamma\delta}\partial_\mu\partial_\alpha\partial_\gamma\bar{\psi}\,\not{s}\,\partial_\delta\partial_\beta\partial_\nu\psi \tag{2.13}$$

and vanishes identically. This shows that the term in (2.7) does indeed generate the correct current, or, in other words, that the Dirac current (2.6) is real.

2.3 Path-integral derivation

In the following we will use the fact that the generating functional in equation (2.5) is gauge invariant, i.e.

$$Z[s] = Z[s^g], \tag{2.14}$$

where under a gauge transformation the source transforms according to

$$s_\mu \to s^g_\mu = g^{-1} \star s_\mu \star g + g^{-1} \star \partial_\mu g\,. \tag{2.15}$$

This invariance follows from the invariance of the measure under local transformations of the fermion fields, i.e. transformations $\psi \to e_\star^{i\alpha} \star \psi = g \star \psi$, together with $\bar{\psi} \to \bar{\psi} \star e_\star^{-i\alpha} = \bar{\psi} \star g^{-1}$. From (2.14) we have:

$$Z[s] = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}g\, e^{-\int \bar{\psi}\star(i\not{\partial}+\not{s}^g)\star\psi} = \int \mathcal{D}g\,\det{}_\star(i\not{\partial} + \not{s}^g)\,, \tag{2.16}$$

where the last equality is obtained after integrating out fermions. Note that the determinant is evaluated with respect to the $\star$-product (we will shortly use the fact that this determinant was computed in [19]). Introduce the connection

$$b_\mu = s^g_\mu\,, \tag{2.17}$$

such that the field strengths of b_μ and s_μ are related according to:

$$f_{\mu\nu}[b] = g^{-1} \star f_{\mu\nu}[s] \star g\,. \tag{2.18}$$

We will choose a gauge where $b_+ = s_+$, with Δ_{FP} being the corresponding Faddeev-Popov determinant (we have $\Delta_{FP} = \det_\star D_+[s_+]$, where $D_+ = \partial_+ + i[s_+,]_\star$). This allows us to write (2.16) in the form,

$$Z[s] = \int \mathcal{D}b_\mu\,\det{}_\star(i\not{\partial} + \not{b})\,\delta[\epsilon_{\mu\nu}(f_{\mu\nu}[b] - f_{\mu\nu}[s])]\,\delta[b_+ - s_+]\,\Delta_{FP}\,. \tag{2.19}$$

By introducing a Lagrange-multiplier field $\hat{a}$ that lives in the "noncommutative" U(1) group with gauge transformation $\hat{a} \to \hat{a}^g = g^{-1} \star \hat{a} \star g$ one can write,

$$Z[s] = \int \mathcal{D}b_\mu\mathcal{D}\hat{a}\,\det{}_\star(i\not{\partial} + \not{b})\; e^{\xi \int \hat{a}\star(f_{\mu\nu}[b]-f_{\mu\nu}[s])}\,\delta[b_+ - s_+]\,\Delta_{FP}\,, \tag{2.20}$$

where ξ is a constant which will be conveniently determined later. Now, make the following change of variables:

$$\begin{aligned}
s_+ &= i\tilde{s}^{-1} \star \partial_+ \tilde{s}\,, \\
s_- &= is \star \partial_- s^{-1}\,, \\
b_+ &= i(\tilde{b} \star \tilde{s})^{-1} \star \partial_+ (\tilde{b} \star \tilde{s})\,, \\
b_- &= (s \star b) * \partial_- (b^{-1} \star s^{-1})\,.
\end{aligned} \tag{2.21}$$

As we stated before, the fermion determinant for the noncommutative U(1) theory has been calculated in [19] with the result that it is:

$$\det{}_\star(i\partial\!\!\!/ + a\!\!\!/) = \exp S_{WZW}[h \star g]\,, \tag{2.22}$$

where $a_+ = h^{-1} \star \partial_+ h$ and $a_- = g \star \partial_- g^{-1}$. The action for the noncommutative WZW model on the r.h.s. is given in equation (2.1). With this result we can express the fermion determinant in terms of the variables in (2.21):

$$\det{}_\star(i\partial\!\!\!/ + b\!\!\!/) = \exp S_{WZW}[\tilde{b} \star \tilde{s} \star s \star b]\,. \tag{2.23}$$

The jacobian for the change of variables $(b_+, b_-) \to (b, \tilde{b})$ gives

$$\mathcal{D}b_+\mathcal{D}b_- = \det{}_\star D_+[\tilde{b} \star \tilde{s}] \det{}_\star D_-[s \star b] \mathcal{D}b\mathcal{D}\tilde{b} = \exp\left(\eta S_{WZW}[\tilde{b} \star \tilde{s} \star s \star b]\right) \mathcal{D}b\mathcal{D}\tilde{b}\,, \tag{2.24}$$

where we recall that the covariant derivatives $D_\pm$ are now in the adjoint representation. Therefore the result for their determinant is the same as for the fundamental representation but with an extra factor, η, that accounts for the change in representation [24]. This factor can actually be computed to be related to the Casimir in the commutative case, but as we shall never need it we simply leave it as η. Furthermore, with this change of variables, one can write the δ-function in (2.20) as:

$$\delta[b_+ - s_+] = \frac{1}{\det_\star D_+[s_+]} \delta\left[\tilde{b} - 1\right]. \tag{2.25}$$

Combining these two results one obtains,

$$\begin{aligned}
Z[s] = \int \mathcal{D}b\mathcal{D}\tilde{b}\mathcal{D}\hat{a} \exp\left((1+\eta) S_{WZW}[\tilde{b} \star \tilde{s} \star s \star b]\right) \times \\
\times \exp\left[\xi \int d^2x\, \hat{a} \star (f_{+-}[b] - f_{+-}[s])\right] \delta[\tilde{b} - 1]\,.
\end{aligned} \tag{2.26}$$

In the gauge $b_+ = s_+$ we have $f_{+-}[b] - f_{+-}[s] = D_+[s_+] \star (b_- - s_-)$ and so this gives,

$$\begin{aligned}
Z[s] = \int \mathcal{D}b\mathcal{D}\hat{a} \exp\left((1+\eta) S_{WZW}[\tilde{s} \star s \star b]\right) \times \\
\times \exp\left[-\xi \int d^2x\, (D_+[s_+] \star \hat{a}) \star \left(is \star b \star \partial_- b^{-1} \star s^{-1}\right)\right].
\end{aligned} \tag{2.27}$$

Note that the expression for the generating functional (2.5) is gauge invariant, the transformation laws for $s, \tilde{s}$ being $\tilde{s} \to \tilde{s} \star g$ and $s \to g^{-1} \star s$. One further change of variables, from $\hat{a}$ to a group valued variable a is defined as follows: $D_+[\tilde{s}] \star \hat{a} = i\tilde{s}^{-1} \star (a^{-1} \star \partial_+ a) \star \tilde{s}$ (note that a is the bose field equivalent to the original fermi field and will be invariant under gauge transformations). The jacobian for the change of variables from $\hat{a}$ to a,[1]

$$\frac{\det_\star D_+[a \star \tilde{s}]}{\det_\star D_+[\tilde{s}]}, \tag{2.28}$$

is, however, not gauge invariant. The trick [13] is to use instead the following jacobian obtained from the above by multiplying with (formally) one:

$$\frac{\det_\star D_+[a \star \tilde{s}]}{\det_\star D_+[\tilde{s}]} \frac{\det_\star D_-[s]}{\det_\star D_-[s]} = \exp\left(\eta S_{WZW}[a \star \tilde{s} \star s]\right) \exp(-\eta S_{WZW}[\tilde{s} \star s]) . \tag{2.29}$$

From this one finally obtains,

$$\begin{aligned} Z[s] = \int \mathcal{D}a\mathcal{D}b \exp\Big((1+\eta)S_{WZW}[\tilde{s} \star s \star b] + \eta S_{WZW}[a \star \tilde{s} \star s] - \eta S_{WZW}[\tilde{s} \star s] + \\ + \xi \int d^2x\, \tilde{s}^{-1} \star a^{-1} \star \partial_+ a \star \tilde{s} \star s \star b \star \partial_- b^{-1} \star s^{-1}\Big) . \end{aligned} \tag{2.30}$$

We can now apply the Polyakov-Wiegmann identity (2.2) for the noncommutative WZW model. Also we choose the up to now arbitrary value of ξ to be $\xi = -\frac{1}{4\pi}(1+\eta)$. This gives the following result for the partition function:

$$Z[s] = \int \mathcal{D}a\mathcal{D}b \exp\Big((1+\eta)S_{WZW}[a\star\tilde{s}\star s\star b] - S_{WZW}[a\star\tilde{s}\star s] + S_{WZW}[\tilde{s}\star s]\Big) . \tag{2.31}$$

Now, make a change of variables $b \to \hat{b} = a \star \tilde{s} \star s \star b$ with trivial jacobian. Then the $\mathcal{D}\hat{b}$-integration factors and it is just a normalization contribution, so that one obtains the simpler expression,

$$Z[s] = \int \mathcal{D}a \exp\left(-S_{WZW}[a \star \tilde{s} \star s] + S_{WZW}[\tilde{s} \star s]\right) . \tag{2.32}$$

One final change of variables with trivial jacobian is $a\star\tilde{s}\star s \to \tilde{s}\star a\star s$; together with the Polyakov-Wiegmann identity this leads to our final result for the bosonization of the free fermion action in equation (2.5) (renaming a as g):

$$\begin{aligned} Z[s] = \int \mathcal{D}g \exp\Big[-S_{WZW}[g]_\star - \frac{1}{4\pi}\int d^2x \Big(s_+ \star s_- - s_+ \star g \star s_- \star g^{-1} - \\ - ig^{-1} \star \partial_+ g \star s_- - is_+ \star g \star \partial_- g^{-1}\Big)\Big] . \end{aligned} \tag{2.33}$$

[1]We denote $D_+[v]f = \partial_+ f + i[v^{-1}\partial_+ v, f]$ and $D_-[v]f = \partial_- f + i[v\partial_- v^{-1}, f]$.

Our prescription for the noncommutative currents becomes:

$$\begin{aligned}\bar\psi\star\gamma_+\psi &\to \frac{i}{4\pi}g^{-1}\star\partial_+ g\,,\\ \bar\psi\star\gamma_-\psi &\to \frac{i}{4\pi}g\star\partial_- g^{-1}\,.\end{aligned}\tag{2.34}$$

This derivation shows that — in the abelian case — the free fermion action on $\mathbb{R}^2_\theta$ is bosonized to the noncommutative U(1) WZW model. From this result it also follows what should be done for the non-abelian system. In the non-abelian case, g in equation (2.1) belongs to the "noncommutative" U(N) group, i.e. $g = e_\star^{i\alpha^a T^a}$ and one should therefore include the ordinary trace over the $N \times N$ matrices in the appropriate places, as in (2.2). When this is done one immediately realizes that the above derivation goes through without any changes, except of course that the group elements now belong to U(N) (note that the evaluation of the noncommutative fermion determinant in [19] also applies to the U(N) case). This shows that in the non-abelian case the free fermionic theory is dual to a noncommutative U(N) WZW model.

With these results in hand, one is lead to make the following observation. The free fermionic action (in both abelian and non-abelian cases) in the noncommutative plane is the same as in the commutative plane, due to its quadratic nature. On the other hand, the standard commutative free fermionic action is equivalent to the commutative WZW model [15], and in the abelian case in particular it is equivalent to a theory of a free scalar field. This shows an equivalence between commutative and noncommutative WZW models, and the map between these two models might be some version of the Seiberg-Witten map [4] between ordinary Yang-Mills theory and noncommutative Yang-Mills theory. It would be very interesting to further explore and make more precise this relation.

3. Noncommutative S-duality

In this section we discuss a noncommutative version of the well-known duality [9] between the Sine-Gordon model and the massive Thirring model. As we have just seen, the free fermion theory discussed in section 2 is dual to a noncommutative WZW model. The next step is to study an interacting fermionic system, where the natural candidate is the Thirring model. We shall see that we can dualize this theory in a straightforward manner, given the results of the previous section. Moreover, we will unravel a strong/weak coupling duality in the procedure.

3.1 The Thirring model

We consider the Thirring model with the usual quartic coupling:

$$S_\lambda = -\frac{\lambda}{2}\int d^2x(\bar\psi\star\gamma^\mu\psi)\star(\bar\psi\star\gamma_\mu\psi) = -\frac{\lambda}{2}\int d^2x\, j^\mu\star j_\mu\,.\tag{3.1}$$

In order to bosonize this term one can either use the bosonization prescription directly (as we know how to bosonize the currents) or do it via a "completing the square" type of prescription at the path-integral level. The two methods obviously yield the same result and we simply use the first. Using the bosonization recipe of section 2 one immediately obtains from equation (2.34) that the four-fermion interaction (3.1) corresponds to the following term in the bose theory:

$$-\frac{1}{2}\lambda\int d^2x\, 2(\bar{\psi}\star\gamma_+\psi)\star(\bar{\psi}\star\gamma_-\psi)\rightarrow\frac{\lambda}{16\pi^2}\int d^2x\, g^{-1}\star\partial_+ g\star g\star\partial_- g^{-1}\,. \tag{3.2}$$

This term is quartic and cannot be made quadratic. This is unlike the commutative theory, where the bosonized current coupling term becomes an extra contribution to the quadratic kinetic term in the scalar action. In here, by including the quartic coupling (3.1) the resulting noncommutative theory becomes significantly different from the corresponding commutative theory by the introduction of an infinite series of derivative terms in the $\star$-product. The bosonized lagrangian therefore becomes:

$$S_{WZW}+\frac{\lambda}{16\pi^2}\int d^2x\; g^{-1}\star\partial_+ g\star g\star\partial_- g^{-1}\,. \tag{3.3}$$

To identify the bosonized fermion theory with a certain bose theory, one still needs a canonically-normalized scalar variable [12]. Recall that we are dealing with noncommutative U(1) group elements, so that one needs to look at scalar fields $\Lambda(x)$ appearing as $g(x)=e_\star^{i\Lambda(x)}$. From the WZW action (2.1) one has the term,

$$\frac{1}{4\pi}\int d^2x\,\partial_+\Lambda\partial_-\Lambda+\cdots\,, \tag{3.4}$$

while from the bosonized Thirring coupling (3.2) we get a term,

$$\frac{\lambda}{16\pi^2}\int d^2x\,\partial_+\Lambda\partial_-\Lambda+\cdots\,. \tag{3.5}$$

By adding these two contributions we get the following kinetic term for the scalar field:

$$\frac{1}{4\pi}\left(1+\frac{\lambda}{4\pi}\right)\int d^2x\,\partial_+\Lambda\partial_-\Lambda\,, \tag{3.6}$$

and so the canonically normalized scalar variable is:

$$\phi=\left[\frac{1}{4\pi}\left(1+\frac{\lambda}{4\pi}\right)\right]^{1/2}\Lambda\,. \tag{3.7}$$

In particular, stability of the bosonic theory requires $\lambda>-4\pi$. This result (3.7) will shortly turn out to be important in determining the relation between the couplings of the noncommutative "Sine-Gordon"[2] and Thirring models.

[2]Observe that we will not actually have a simple noncommutative extension of the Sine-Gordon model due to the extra term appearing in (3.3).

JHEP07(2000)030

3.2 The massive thirring model

We shall now turn to the fermion mass coupling. The relevant term is

$$S_m = m \int d^2x \, \bar{\psi} \star \psi \,. \tag{3.8}$$

In order to extend the previous discussion to the massive Thirring model we follow the procedure outlined in [12], as it applies to our situation. Indeed one can bosonize the mass term by considerations of chiral symmetry alone, and the discussion in [12] directly applies in here as well (as one is considering global chiral symmetry for which the $\star$-product collapses to the standard product).

Free fermions are invariant under the global $\mathrm{U}_A(1)$ axial symmetry $\psi \to e^{i\alpha\gamma_5}\,\psi$ and $\bar{\psi} \to \bar{\psi}\, e^{i\alpha\gamma_5}$, and one expects that the bosonic theory will share such a symmetry [15]. On the other hand, under the duality procedure of the previous section (see also [12]) one is gauging the global vector symmetry of the free fermions and the axial symmetry will not survive quantization due to the presence of the background gauge field. This axial anomaly in the noncommutative plane was computed in [19],

$$\partial_\mu j_5^\mu = -\frac{1}{2\pi}\epsilon^{\mu\nu}\hat{F}_{\mu\nu} \,, \tag{3.9}$$

where the axial current is $j_5^\mu = \bar{\psi} \star \gamma^\mu\gamma^5\psi$ and $\hat{F}_{\mu\nu}$ is the noncommutative gauge field strength. Because both initial and final theories (under duality) share the same symmetry, it may seem odd that this symmetry is broken under the duality procedure. The solution to make it manifest throughout the dualization procedure is to include a transformation on the Lagrange multiplier as well [12]. Indeed, recall that the Lagrange multiplier field Λ appears in the initial gauged action as,

$$\sim \exp\left(\int d^2x \, \frac{1}{2\pi}\Lambda \star \epsilon^{\mu\nu}\hat{F}_{\mu\nu}\right) . \tag{3.10}$$

The observation is that if the field Λ transforms as $\Lambda \to \Lambda - \alpha$ under axial transformations, then the Lagrange multiplier term (3.10) will cancel the axial anomaly term (3.9) as it appears in the path-integral, and chiral symmetry is made manifest. In summary, what we have done is to nail down what is the correct transformation of the new field Λ under chiral rotations, and this is uniquely defined by the previous considerations (for details see [12]).

The bottom line is that one can now proceed to deduce the bosonization of fermionic mass terms from these transformation properties. Indeed, in the presence of a mass term the fermi theory is no longer axial symmetric, but the axial transformation rules nevertheless remain the same. As we shall see in the following, these rules alone are enough information for a unique determination of the bosonized composite operator that corresponds to the fermionic quadratic term. For instance, we would like to bosonize a term as $\psi_R^\dagger \star \psi_L$, which amounts to finding an appropriate bosonic functional $\mathcal{F}(\Lambda)$ such that $\mathcal{F}(\Lambda) \equiv \psi_R^\dagger \star \psi_L$. Under global chiral transformations the chiral mass term transforms as,

$$\psi_R^\dagger \star \psi_L \to e^{-2i\alpha} \psi_R^\dagger \star \psi_L \,, \tag{3.11}$$

and due to the Λ chiral transformation rule just deduced, it follows that the bosonic functional must satisfy,

$$\mathcal{F}(\Lambda - \alpha) = e^{-2i\alpha} \mathcal{F}(\Lambda) \,, \tag{3.12}$$

where we recall that due to the global character of the rotation, the exponential involves no $\star$-product, even though the functional $\mathcal{F}$ will be defined in terms of the field Λ through a $\star$-product. Indeed, it immediately follows that

$$\mathcal{F}(\Lambda) \propto e_\star^{2i\Lambda} \tag{3.13}$$

uniquely solves the functional equation (where the exponential is properly defined with the $\star$-product). This whole procedure naturally follows from [12] as one is dealing with global chiral rotations. What all this amounts to, is that the mass term (3.8) bosonizes to:

$$S_m = \int d^2x \, m\alpha_0 \cos_\star 2\Lambda \,, \tag{3.14}$$

where α_0 is a constant associated to the zero point energy (as in the commutative case [9]), and the noncommutative cosine is defined naturally by $\cos_\star \varphi = \frac{1}{2}(e_\star^{i\varphi} + e_\star^{-i\varphi})$. This story is very similar to the commutative Sine-Gordon/Thirring duality. In particular, the coupling constant in the bosonic theory is defined as β and appears in the action exactly through a cosine potential. In the noncommutative case, this would be,

$$\int d^2x \, \alpha \cos_\star \beta\phi \tag{3.15}$$

where the field ϕ is the canonically normalized field. Finally, since the canonically normalized field was $\phi = \sqrt{\frac{1}{4\pi}(1 + \lambda/4\pi)}\Lambda$ we obtain the following S-duality property:

$$\frac{16\pi}{\beta^2} = 1 + \frac{\lambda}{4\pi} \,. \tag{3.16}$$

This is exactly the same type of relation as that of the commutative theory (1.3). At first glance, one could worry that when we plug this relation (3.16) back in (3.3) there could be terms going like inverse powers of β which would spoil the strong/weak coupling duality. However, this does not happen as one should also recall that the scalar fields have to be canonically normalized according to (3.7). What therefore happens is that an expansion of (3.3) in powers of the canonically normalized field ϕ will only produce interaction terms proportional to positive powers of the bosonic coupling constant β. From this we see that the strong/weak coupling duality between the Sine-Gordon model and the massive Thirring model survives on a noncommutative space, as we have just shown.

It is known that the T-duality of string theory can be interpreted as Morita equivalence in noncommutative geometry [25, 26]. One can wonder if there could be a similar geometrical interpretation of string theory S-duality in terms of noncommutative geometry. In here we have given some first steps, by showing that one can construct quantum field theoretical models in noncommutative space displaying S-duality.

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JHEP07(2000)030